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Čerenkov radiation by a particle moving with a non-uniform velocity

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Abstract. Energy loss by a charged particle moving with non-uniform velocity in a uniform dielectric medium is investigated. Assuming constant deceleration, an expression is derived for radiation emitted in an arbitrary distance x, and the case of Čerenkov radiation is discussed in detail. Expressions are presented for energy radiated per unit frequency interval and for Čerenkov angle containing first-order terms in retardation.

1. Introduction

Čerenkov radiation is emitted when a charged particle moves in a dielectric medium with a velocity greater than the phase velocity of electromagnetic radiation in the medium. In the analysis of Čerenkov radiation it is generally assumed that the particle moves with a uniform velocity (Marmier and Sheldon 1969). This assumption is in contradiction with the basic conservation laws. Čerenkov radiation in a medium arises due to the time history of the asymmetric electric polarization produced by the fast moving particle in the medium. This asymmetric polarization, in turn, retards the motion of the charge which must slow down as it moves in the medium. Zin (1961) and Sen Gupta (1965) have emphasized from energy considerations that the assumption of uniform velocity is not justified. In fact the electromagnetic field energy in Čerenkov radiation appears at the cost of the kinetic energy of the moving particle. The retardation of a charge emitting Čerenkov radiation is of the order $(e^2\omega^2/8\pi\epsilon_0 mc^2)(1-c^2/n^2v^2)$, where m and v are the mass and velocity of the particle, n is the refractive index of the medium, and ω is the frequency of the radiation. For light particles the magnitude of this retardation is large and should be taken into account.

In this paper we investigate the energy loss per unit frequency interval by a particle moving with a non-uniform velocity in a uniform dielectric medium. The result is applied to Čerenkov radiation when the charge moves with velocity greater than the phase velocity of electromagnetic waves in the medium. For an exact analysis of the problem, the retardation must be found from self-consistency considerations. However, this makes the problem very involved and complex. In most of the practical applications, the path available to the charge is not very large and hence it is plausible to assume a constant average retardation. In this paper we calculate Čerenkov energy loss in a distance x of the path of the particle taking first-order terms in retardation into account. The contribution of the retardation to the Čerenkov angle is also calculated to first order.

2. Energy loss by the moving particle

Consider the motion of a charge e moving with a velocity $\boldsymbol{v}$ and a constant retardation $\boldsymbol{a}$ along the x axis in a uniform medium for a time T. The current density $\boldsymbol{J}$ due to this will be

$$\boldsymbol{J}(\boldsymbol{r}', t) = e(\boldsymbol{v}-\boldsymbol{a}t)\,\delta(x'-vt+\tfrac{1}{2}at^2)\,\delta(y')\,\delta(z')f(t) \tag{2.1}$$

where $f(t)$ is defined by

$$f(t) = \begin{cases} 1 & \text{for } 0 \leqslant t \leqslant T \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

The Fourier transform of the current density is

$$\boldsymbol{J}_\omega(\boldsymbol{r}') = \frac{\hat{\boldsymbol{x}}'ae}{\pi}\,\delta(y')\,\delta(z')\int_0^T (v-at)[\delta(a^2t^2-2avt+2x'a)]\exp(\mathrm{i}\omega t)\,\mathrm{d}t \tag{2.3}$$

which can be simplified to

$$\boldsymbol{J}_\omega(\boldsymbol{r}') = \frac{\hat{\boldsymbol{x}}'e}{2\pi l}\,\delta(y')\,\delta(z')\int_{v-aT}^{v} s[\delta(s-1)+\delta(s+1)]\exp\left(\mathrm{i}\frac{\omega}{a}(v-s)\right)\mathrm{d}s \tag{2.4}$$

where

$$v-at = s \qquad \text{and} \qquad v^2-2ax' = l^2. \tag{2.5}$$

Since l is positive and lies between $v-aT$ and v, only $\delta(s-1)$ contributes to the integral for $\boldsymbol{J}_\omega$. Hence we get

$$\boldsymbol{J}_\omega(\boldsymbol{r}') = \frac{\hat{\boldsymbol{x}}'e}{2\pi}\,\delta(y')\,\delta(z')\exp\left(\mathrm{i}\frac{\omega}{a}(v-1)\right). \tag{2.6}$$

The energy radiated per unit solid angle in a frequency band $\mathrm{d}\omega$ is given by Panofsky and Phillips (1964) as

$$\frac{\mathrm{d}U_\omega}{\mathrm{d}\Omega}\mathrm{d}\omega = \frac{\mathrm{d}\omega}{4\pi\epsilon_0 cn}\left|\int J_\omega k\sin\theta\exp(-\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{x}')\,\mathrm{d}v'\right|^2 \tag{2.7}$$

where θ is the angle between wavevector $\boldsymbol{k}$ and $\boldsymbol{J}$; $\boldsymbol{x}'$ is the position vector of the particle.

From equations (2.7) and (2.6) we obtain the radiation energy loss $U_\omega\,\mathrm{d}\omega$ in frequency band $\mathrm{d}\omega$ as

$$U_\omega\,\mathrm{d}\omega = \frac{e^2\omega^2 n\,\mathrm{d}\omega}{16\pi^3\epsilon_0 c^3}\int I\sin^2\theta\,\mathrm{d}\Omega \tag{2.8}$$

where

$$I = \left|\int_0^x \exp\left\{\mathrm{i}\left[\left(\frac{\omega}{a}\right)(v-1)-kx'\cos\theta\right]\right\}\mathrm{d}x'\right|^2 \tag{2.9}$$

and

$$k = \frac{n\omega}{c}.$$

Since we are interested in the energy radiated by the particle from time $t = 0$ to $t = T$, this corresponds to the path available to the particle from 0 to x. Solving the integral (2.9) we get

$$I = \frac{1}{k^2 \cos^2\theta}\left\{\left[\left(\frac{\pi\omega^2}{ak\cos\theta}\right)^{1/2}(C(\beta)-C(\alpha))+(\sin\beta^2-\sin\alpha^2)\right]^2 + \left[\left(\frac{\pi\omega^2}{ak\cos\theta}\right)^{1/2}(S(\beta)-S(\alpha))-(\cos\beta^2-\cos\alpha^2)\right]^2\right\} \quad (2.10)$$

where

$$\alpha = \left(\frac{k\cos\theta}{2a}\right)^{1/2}\left(v-\frac{\omega}{k\cos\theta}\right)$$
$$\beta = \left(\frac{k\cos\theta}{2a}\right)^{1/2}\left((v^2-2ax)^{1/2}-\frac{\omega}{k\cos\theta}\right). \quad (2.11)$$

$C(\alpha)$ and $S(\alpha)$ are the Fresnel integrals (Magnus and Oberhettinger 1949).

3. Čerenkov radiation

For a particle moving with velocity greater than the phase velocity of electromagnetic waves in the medium, both α and β are positive. Using asymptotic expansions for the Fresnel integrals and keeping only first-order terms in a in the expansion, we obtain

$$I = 4k^2\cos^2\theta\sin^2\left[\frac{x}{2}\left(k\cos\theta-\frac{\omega}{v}-\frac{\omega ax}{2v^3}\right)\right]\left(k\cos\theta-\frac{\omega}{v}-\frac{\omega ax}{2v^3}\right)^{-2}. \quad (3.1)$$

Thus equation (2.9) can be written as

$$U_\omega\,\mathrm{d}\omega = \frac{e^2\omega^2 n\,\mathrm{d}\omega}{2\pi^2\epsilon_0 c^3}\int_{-1}^{+1}\sin^2\theta\sin^2\left[\frac{x}{2}\left(k\cos\theta-\frac{\omega}{v}-\frac{\omega ax}{2v^3}\right)\right] \times\left(k\cos\theta-\frac{\omega}{v}-\frac{\omega ax}{2v^3}\right)^{-2}\mathrm{d}(\cos\theta). \quad (3.2)$$

The angular integral has the characteristics of a δ function. This shows that the radiation has a fairly sharp maximum at the angle θ given by

$$\cos\theta = \frac{\omega}{kv}+\frac{\omega ax}{2kv^3}. \quad (3.3)$$

Evaluating the integral (3.2), we get

$$U_\omega\,\mathrm{d}\omega = \frac{e^2 x}{4\pi\epsilon_0 c^2}\left(1-\frac{c^2}{n^2v^2}-\frac{c^2ax}{n^2v^4}\right)\omega\,\mathrm{d}\omega. \quad (3.4)$$

4. Conclusions

In this paper we have investigated the effect of deceleration on Čerenkov radiation. It is seen from equation (3.4) that the energy radiated per unit frequency interval per unit path U_ω/x is a function of path and it decreases with the path traversed. Equation (3.3) shows that the Čerenkov angle θ also decreases with increasing path, as it should.

The approximate value for the retardation can be obtained by equating the radiation emitted per second to the rate of kinetic energy loss of the particle. Though the retardation of the particle is not constant, however, the assumption of a constant retardation is justified because the time of flight available to the particle is generally not very large and an average value of retardation can be used in calculating the Čerenkov energy loss and Čerenkov angle.

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